Relation between fractal dimension and roughness index for fractal surfaces

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This paper discusses the relation between fractal dimension and roughness index for fractal surfaces of solids. The applicability of the relation to fracture of Mode III+Mode I complex loading is shown. The applicability to other rough surfaces is discussed. [S1063-651X(99)09210-7]

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I. INTRODUCTION

Since Mandelbrot, Passoja, and Paulay [1] showed that fractured surfaces are fractals in nature and that the fractal dimension of fractured surfaces correlates well with the toughness of materials, one of us [2] has analyzed the critical crack extension force with the fractal model and pointed out that the true areas of the fractured surfaces in materials are actually larger than that calculated from a flat fractured surface. Since then, many authors have done experiments on this problem [2–7]. Now, there seems no point in denying that the fractured surface is a fractal with self-affine property [8]. However, the fractal dimension of even the simplest self-affine fractals is not uniquely defined [9]. The difficulties have been illustrated with the case of fractional Brownian motion by Voss [10] and Pietronero [11]. In his paper [10], Voss showed that $D=2-H$ in the box counting method and $1/H$ in the limiting case of the "coastline" method on smaller scales and 1 on larger scales. Pietronero came to the same conclusion [11] by taking normal Brownian motion as an example. It seems to be already known that the apparent fractal dimension D of a self-affine surface, and the apparent Hurst exponent H of a self-similar surface, might be scale dependent in principle. The questions are as follows: How strong are their scale dependences quantitatively? Why did previous authors obtain nice linear relationships in double logarithmic plots both of $L(\varepsilon)$ vs ε and height vs minimum distance? We need a quantitative analytical relation between D and H in materials. To our knowledge, no analytical expressions for quantitative analysis of experimental results has been done before.

In the case of fractured surfaces, the coastline method seems nearer to a real situation and discussions on limiting cases are not enough to show the detailed quantitative relations between D and H in the entire range of scales. In this paper, we use the coastline method to show the quantitative relations between D and H and show some interesting findings that are not sufficient to be understood by qualitative and approximate estimation. The applicability of the relation to fracture of Mode III+Mode I complex loading is shown in Sec. IV.

II. RELATION OF *D* TO *H* FOR SELF-AFFINE SURFACES

Since most real surfaces scale differently in the plane of fracture and in the vertical direction, it seems that they are self-affine rather than self-similar. What will happen if one measures the self-affine surfaces with D artificially, or describes the self-similar surfaces with H?

If the surface is self-affine, we may determine H from double logarithmic plots of $\Delta V(t)$ vs Δt, where V is the vertical height and t is the horizontal axis. Then, H might be a constant that is independent on the yardstick. On the other hand, if we measure the fractal dimension of the surface artificially, the D value might be yardstick dependent. However, in the earlier works on fracture [1,8], constant D values have been obtained in many cases, even though the concept of self-affine property of fractured surfaces has been accepted generally. Straight lines on double-logarithmic plots both on ΔV vs Δx [12,13] and on $L(\varepsilon)$ vs ε [1,8] have been obtained. The reason why may be explained in the following.

For a coastline, one may divide the curve into N segments by walking a ruler of size l along the curve. The length along each segment is

$$\frac{l}{t_0}=\left[\left(\frac{\Delta t}{t_0}\right)^2+\left(\frac{\Delta V}{t_0}\right)^2\right]^{1/2}. \tag{1}$$

Using the relation

$$\frac{\Delta V_H}{V_0}=\left(\frac{\Delta t}{t_0}\right)^H=N^{-H} \tag{2}$$

and the relations $l_0=\sqrt{2}t_0$, $V_0=t_0$,

$$\frac{\sqrt{2}l}{l_0}=\frac{\Delta t}{t_0}\left[1+\left(\frac{\Delta t}{t_0}\right)^{2H-2}\right]^{1/2}. \tag{3}$$

Then,

$$N\left(\frac{l}{l_0}\right)=\left(\frac{\Delta t}{t_0}\right)^{-1}=\frac{1}{\sqrt{2}}\left(\frac{l}{l_0}\right)^{-1}\left[1+\left(\frac{\Delta t}{t_0}\right)^{2H-2}\right]^{1/2}, \tag{4}$$

$$N\left(\frac{l}{l_0}\right)=\left(\frac{l}{l_0}\right)^{-D}. \tag{5}$$

Therefore,

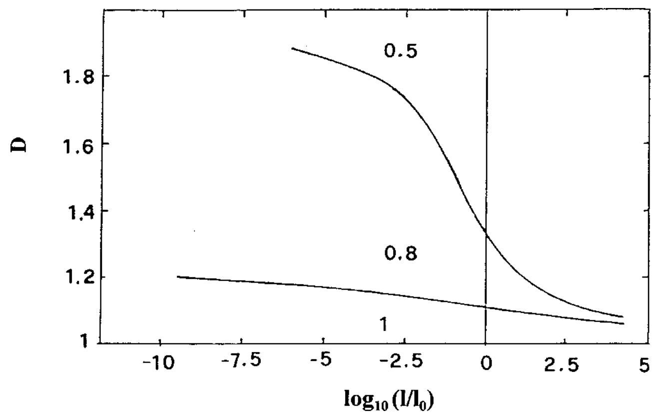

FIG. 1. Relationship of $D(H,\Delta t/t_0)$ with l/l_0.

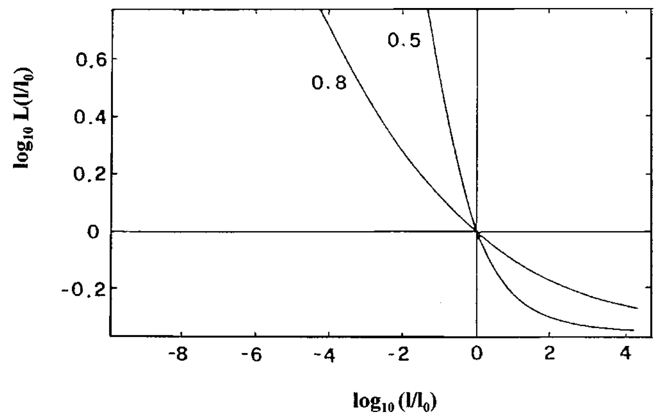

FIG. 2. Double common logarithmic relationship of $L(l/l_0)$ with l/l_0.

$$D=1-\frac{\ln\left[\frac{1}{2}\left(1+\left(\frac{\Delta t}{t_0}\right)^{2H-2}\right)\right]}{2\ln\left(\frac{l}{l_0}\right)};\tag{6}$$

let

$$\xi=\frac{1}{2}\left[1+\left(\frac{\Delta t}{t_0}\right)^{2H-2}\right]$$

and from Eq. (3), we obtain

$$D\left(H,\frac{\Delta t}{t_0}\right)=1-\left[1+\frac{2\ln\left(\frac{\Delta t}{t_0}\right)}{\ln\xi}\right]^{-1}.\tag{7}$$

(i) When $\Delta t/t_0 \ll 1$, $N=(\Delta t/t_0)^{-1}\approx(l/l_0)^{-1/H}$; and then $D=1/H$.

(ii) When $\Delta t/t_0=1$, D approaches the limiting value $2/(1+H)$.

Figure 1 shows the relationship of $D(H,\Delta t/t_0)$ with l/l_0. In the double-logarithm plots, the dependence of D with $\Delta l/l_0$ becomes weaker and weaker as H rises from 0.5 to 1. One cannot judge whether D is dependent on the yardstick when the H value rises up to a certain value near unity (say $0.8<H<1$) within the accuracy of measurements. In this case, we cannot distinguish which, self-similarity or self-affinity, is better to describe the fractal surface. Therefore, in the early works of fracture, nice fractal dimensions have been obtained in previous measurements (for example, see [8]).

Figure 2 shows the relationship of $L(\Delta l/l_0)$ with l/l_0. In this figure, $L(l/l_0)$ is calculated by $(l/l_0)^{1-D}$ where $D(l)$ is calculated by Eq. (7). Because of D a function of H and l, $L(H,l)$ and l are not linear relations in double-logarithmic plots. From Fig. 2, the deviation of the linear relationship is larger when $H=0.5$ and smaller when $H=0.8$. $L(l/l_0)$ is independent of l/l_0 when $H=1$. Similarly, as H approaches 1 (say $0.8<H<1$), we cannot judge whether it is a curve or a straight line within the range of experimental error.

Moreover, the values of $L(l)$ are points on the L vs l double-logarithmic plots with various values of H. The slope of the curve does not have the meaning of fractal dimension. The real fractal dimension $(1-D)$ is the slope of the straight line connected to the point $(L/L_0, l/l_0)$ and the zero point (0,0).

III. RELATION OF H TO D FOR SELF-SIMILAR SURFACES

On the other hand, if the fracture surface is of self-similar structure, one may determine D from the double-logarithmic plots of the $L(l/l_0)$ and l/l_0 relationship. The value of D might be a constant and is independent of l/l_0. If one measures the roughness exponent of the surface, the value of H might be yardstick dependent.

Now, with similar derivation, we have

$$H(l,D)=1+\frac{\ln\left[2\left(\frac{l}{l_0}\right)^{2-2D}-1\right]}{2\ln\left(\frac{\Delta t}{t_0}\right)}.\tag{8}$$

(i) When $l/l_0\ll 1$

$$H(l,D)=\frac{1}{D}+\frac{\ln 2}{2\ln\left(\frac{\Delta t}{t_0}\right)}\approx\frac{1}{D}.\tag{9}$$

(ii) When $l/l_0=1$, the limiting value of $H(l,D)=2/D-1$.

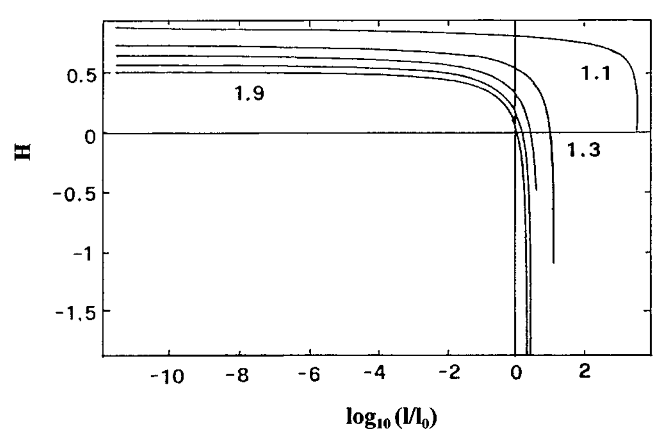

FIG. 3. Relationship of $H(D,l/l_0)$ with l/l_0.

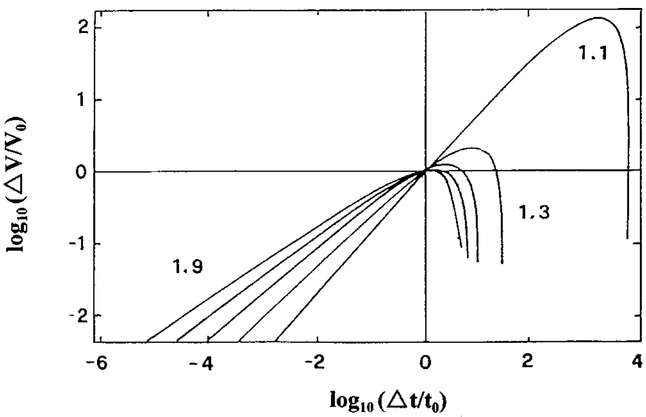

FIG. 4. Double common logarithmic relationship of $\Delta V(D,l/l_0)$ with $\Delta t/t_0$.

Figure 3 shows the double-logarithmic plots of $H(D,l/l_0)$ and l/l_0. Similar to the above, the dependence of H on l/l_0 becomes weaker and weaker as D approaches unity.

Figure 4 shows the double-logarithmic plots of $\Delta V(D,l/l_0)$ and $\Delta t/t_0$. As above, the deviation from a straight line is smaller and smaller when D approaches unity.

IV. THE APPLICABILITY OF THIS RELATIONSHIP TO MODE III+MODE I COMPLEX FRACTURE

It is well known that the fractured surfaces are self-affine under Mode I deformation (see Fig. 6) [12]. Recently, Daguier, Bouchaud, and Lapasset reported that even under mode I loading, the crack front along the plane of crack propagation is not self-affine, but self-similar in the case of quenched disorder of impurities [15]. We would like to remind the reader that in the practical cases, there is a great possibility of the presence of complex mode loading (Mode I+Mode III; Mode I+Mode II and so on). Presence of pure mode I or any other kind of mode loading is rare. If it is under complex mode loading, we need to know whether the surface is mainly self-affine or self-similar. Qualitative understanding is not sufficient to explain these processes. Therefore, it is important to find out the quantitative relationship of them and find a way to distinguish whether the surface is mainly self-similar or self-affine experimentally. Figure 5 is the schematic figure of three modes of deformation in fracture.

Self-similarity implies the presence of ''overhangs'' in the surface structure: to satisfy isotropy, scale invariance of all possible directions should be presented equally. There are

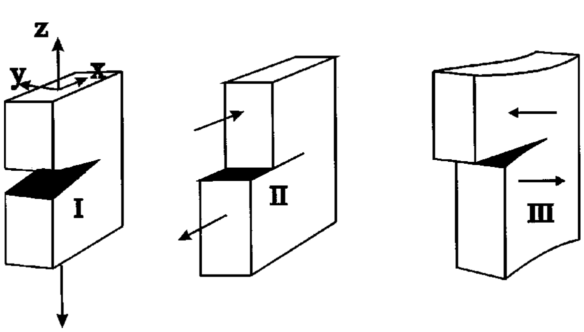

FIG. 5. Modes of deformation in fracture.

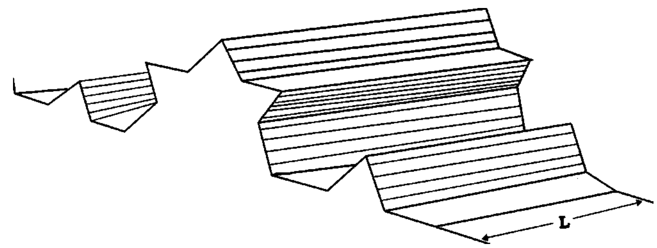

FIG. 6. A triadic Koch surface.

a number of known examples for such surfaces, including coastlines and the surface of silica colloid particles or materials used for catalysis [14]. The perimeter of the island in the slit-island method [1] is like the coastline, the prototype of self-similar scaling fractal. This was verified by the linearity in early experimental measurements of fractal dimensions for the fractured surfaces [1–7]. In Mode II deformation, the type of deformation constructing a manifestly fracture surface for sliding the triadic Koch curve along the x direction perpendicular to the yz plane on which a Koch curve crack line exists is possible (also see Fig. 6). A similar situation may occur on the xz plane in Mode III deformation. These processes do not need to avoid overhangs. Figure 7 is a slant crack in the thickness direction. When $\alpha=0$, it is a pure Mode III loading; when $\alpha=90°$, it is a pure Mode I loading. It is in mixed complex mode cases, when $0<\alpha<90°$, where both Modes I and III loading exist. This is the usual case that the plane of a through crack in laboratory specimens is often tilted at an angle α with the plate surface as shown in Fig. 7. This complex mode of loading leads to an important effect that, for the Poisson ratio $\gamma>0.25$, the minimum applied stress no longer occurs at $\alpha=90°$ but at

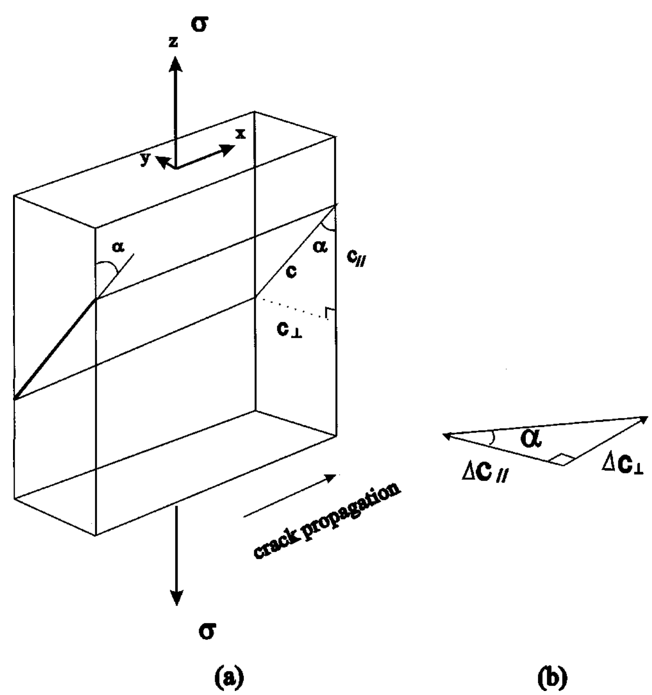

FIG. 7. (a) Schematic figure of Mode I and Mode III mixed deformation crack; (b) roughness $\Delta c_\parallel$ and $\Delta c_\perp$ components (not crack increments) produced by $c_\parallel$ and $c_\perp$, which are two components of crack front, respectively.

angles determined by an arcsin function of γ. Thus, as the crack tilts away from normal to the load or Mode I loading, it becomes less impossible for the specimen to support the applied load [16]. In this case, the crack may not propagate along its original direction; the tilted angle α may change to α'. However, in this paper we do not go into the details of fracture mechanics. We assume α is the instantaneous tilted angle. In the condition that the crack front line of Mode I is self-affine and that of Mode III is self-similar, the crack front line on the *XY* plane of Fig. 7 is a line of mixed self-affinity and self-similarity. Mode I deformation produces the in-plane roughness of the self-affine crack front along the x direction and Mode III deformation produces the in-plane roughness of the self-similar deformation along y, the direction of crack propagation, which is perpendicular to the direction of the Mode III crack front component, z and x.

The relation of $L(H,\Delta c/c_0)$ to $\Delta c/c_0$ is given by the vector sum of its two components:

$$L\left(H,\frac{\Delta c}{c_0}\right)=\left\{\left(\frac{\Delta c_\perp}{c_{\perp 0}}\right)^{2[1-D_\perp(H_\perp,\Delta c_\perp/c_{\perp 0})]}\sin^2\alpha + \left(\frac{\Delta c_\parallel}{c_{\parallel 0}}\right)^{2(1-D_\parallel)}\cos^2\alpha\right\}^{1/2}\sim\left(\frac{\Delta c}{c_0}\right)^{1-D_{\mathrm{eff}}}. \tag{10}$$

The relation of $(\Delta V/V)(D,\Delta c/c_0)$ to $\Delta c/c_0$ is also given by the vector sum of its two components:

$$\frac{\Delta V}{V_0}\left(D,\frac{\Delta c}{c_0}\right)=\left\{\left(\frac{\Delta c_\perp}{c_{\perp 0}}\right)^{2H_\perp}\sin^2\alpha + \left(\frac{\Delta c_\parallel}{c_{\parallel 0}}\right)^{2H_\parallel(D_\parallel,\Delta c_\parallel/c_{\parallel 0})}\cos^2\alpha\right\}^{1/2}\sim\left(\frac{\Delta c}{c_0}\right)^{H_{\mathrm{eff}}}, \tag{11}$$

where c is the length of the crack under mixed mode loading. $\Delta c_\perp$ and $\Delta c_\parallel$ are its roughness components along the x and y directions in the *xy* plane, respectively.

Let $\Delta c/c_0=\Delta c_\perp/c_{\perp 0}=\Delta c_\parallel/c_{\parallel 0}$ for simplicity and assume $H_\perp=0.8$ and $D_\parallel=1.3$ for the case of usual roughness of fractured surfaces. Relationships of Eqs. (10) and (11) are shown in Figs. 8(a) and 8(b), respectively. In Fig. 8(a), the relationship of L vs $\Delta c/c_0$ is linear when $\alpha=0$, which corresponds to the case of Mode III loading (pure self-similar). The relationship deviates from linearity when α increases from 0 to $\pi/2$, which corresponds to the case of Mode III and Mode I mixed loading (self-similar mixed with self-affine). The relationship of L vs $\Delta c/c_0$ shows the maximum deviation from linearity in the case of α equal to $\pi/2$, which corresponds to pure Mode I loading (pure self-affine). In Fig. 8(b), the relationship of $\Delta V/V_0$ vs $\Delta c/c_0$ is linear when $\alpha=\pi/2$, which corresponds to the case of Mode I loading (pure self-affine). The relationship deviates from linearity when α decreases from $\pi/2$ to 0, which corresponds to the case of mode I and mode III mixed loading (self-affine mixed with self-similar). The curve of $\Delta V/V_0$ vs $\Delta c/c_0$ shows the maximum deviation from linearity in the case of α equals to 0, which corresponds to pure Mode III loading (pure self-similar). Moreover, from Fig. 8(a), we may estimate the approximate slope of the curve in the case of $\pi/2$ at the small scale range. It is $D=1.226$, which is just between 1.2 $(2-H)$ and 1.25 $(1/H)$. From Fig. 8(b), we may also estimate the approximate slope of the curve in the case of $\alpha=0$ at the small scale range. It is 0.727, which is just between 0.7 $(2-D)$ and 0.769 $(1/D)$. It seems that either $D+H=2$ or $DH=1$ is really a rough estimation as pointed out by Peitgen, Jurgens, and Saupe [17] earlier.

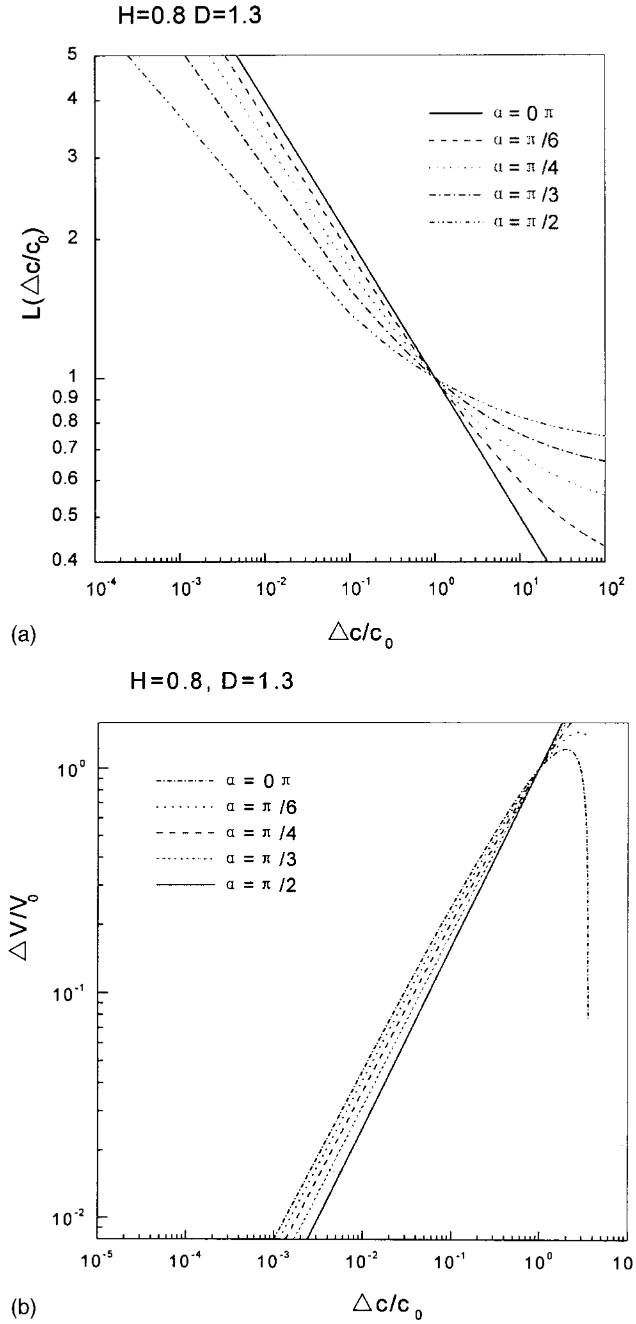

FIG. 8. (a) $L(\Delta c/c_0)$ vs $\Delta c/c_0$ relationship; (b) $\Delta V/V_0$ vs $\Delta c/c_0$ relationship.

V. SUMMARY

From the above analysis, one may draw the following conclusions. If one describes a surface of a self-affine structure with fractal dimension the apparent fractal dimension might be yardstick dependent. However, the dependence cannot be distinguished as the H value is near unity. On the other hand, if one describes a surface of self-similar structure, with roughness exponent, the apparent H value might be yardstick dependent. However, the dependence cannot be correctly appraised when the D value is near unity.

In principle, comparing the linearity of the H vs l and the D vs l relation in double-logarithmic plots, one may make an appraisal as to which structure, either self-affine or self-similar, is the real one. However, if the surface appears to flatten, this experimental method is not sensitive; one should then adopt other experimental methods, say, direct observations by means of scanning electron microscopy, scanning tunneling microscope, etc. to make the appraisal.

In addition, comparing Figs. 1 and 3, we can see that the dependence of $H(D,l/l_0)$ on l/l_0 is weaker than that of $D(H,l/l_0)$ on l/l_0. Then, the measurement of the roughness exponent is a less sensitive way to judge the deviation from self-affinity than the fractal dimension to the deviation from self-similarity. The range of H values from 0.5 to 1 is half the range of D values from 1 to 2. Using the measured values of H parameters to characterize the roughness of materials, the differences among them are easy to be ignored or the universal properties are easy to be exaggerated.

Moreover, for judging a structure to fractal or nonfractal, we recommend measuring both the double-logarithmic relations of $L(\varepsilon)$ vs ε and of ΔV vs Δx. If you find that D is scale dependent, perhaps it is still a fractal of self-affinity. On the other hand, if you find that H is scale dependent, perhaps it is still a fractal of self-similarity. One may remember that the measured values of H parameters is less sensitive; an approximate straight line in Fig. 4 is not rigorous enough to judge its self-affine property. Comparing with Fig. 4, Fig. 2 is more conclusive, because the nonlinear behavior in Fig. 2 shows that it is not a fractal of self-similar definitely.

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